# Steady-State Reliability Analysis of Repairable Systems Subject to System Modifications

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*Environmentally conscious design is approached through analysis and further development of a reliability model that facilitates design for reuse of products. Many reliability models may not be suitable for describing systems that undergo repairs performed during remanufacture and maintenance since they do not allow for the possibility of system reconfiguration. In this paper, expressions of reliability indices of a model that allows system modifications during repair are derived. These reliability indices that describe a population of repairable systems are theoretically proven to reach steady state, supporting the simulation results of the model. This model can be used to estimate life-cycle replacement requirements for systems that are remanufactured, thereby facilitating decisions during system design and use. An example illustrates the application of the model to a relevant industry.*

## 1 Introduction

This section begins with the motivation for developing a reliability model that allows product modification during repairs performed during remanufacture or maintenance. Next presented is a brief summary of previous work towards this goal. Finally, the objectives of this paper are identified.

**1.1 Motivation.** The motivation for this work originated during study of product design to facilitate remanufacture. Remanufacture consists of the production-batch disassembly, cleaning, replacement and refurbishment of worn parts of defective or obsolete products. Remanufacture can be considered as recycling at the component level instead of the material level. Therefore, remanufacture avoids the possibly unnecessary resource consumption of scrap-material recycling while preserving the value added to a component during manufacture (Lund, 1983). The production-batch and off-site nature of remanufacture results in a labor cost significantly lower than that required for individual repair. For appropriate products, remanufacture offers significant economic and ecological advantages over other end-of-life options.

Product design that facilitates any of the steps in remanufacture, such as disassembly, cleaning, refurbishment, etc., is beneficial to remanufacture. However, part reuse is the main goal of remanufacture. That is, the ease of disassembly and cleaning is not relevant unless most parts can be reused either with or without refurbishment. Refurbishment aims to return a part to a like-new or better condition. When parts are to be reused, the reliability of the part and of the product is very important.

During remanufacture, parts can be replaced by the same or different types of parts, either new or refurbished. Therefore, the failure probability density function of the replacement part may be different from that of the original part. The rest of the product either remains unchanged or is reconfigured to accommodate the replacement parts.

**1.2 Previous Work.** The possibility of system modification is not included in most reliability models for repairable systems. A system can be defined as a collection of two or more parts designed to perform one or more functions. A repairable system is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to satisfactory performance by any method other than replacement of the entire system. Ascher and Feingold (1984), who surveyed the literature on reliability of repairable systems, observed that much of the work models one of two extremes. The first extreme represents the repair process as returning a system to a same-as-new condition. The other extreme assumes that the repair returns the system to a level at which it was operating before failure. Such a model is referred to as a minimal repair model (Barlow and Hunter, 1960). Moderation of the above extremes includes the following. Brown and Proschan (1980) model imperfect repair, where renewal to a same-as-new state occurs with a probability. Nakagawa (1980) models partial renewal of a system at maintenance times.

To better portray remanufacture, a reliability model suitable for describing a population of repairable systems subject to system modifications was developed by Shu and Flowers (1998). This model allows replacement of failed parts with both the same and different types of parts. The effects of the reliability of original and replacement parts on life-cycle replacement cost were considered to explore alternatives in initial part designs and remanufacture process plans. The life-cycle replacement cost is closely related to reliability indices such as replacement rate and average age of a population of systems or parts. Simulation of this model shows that these indices exhibit a transient behavior and eventually reach steady state. To better understand the simulation results, a theoretical analysis is required.

**1.3 Objectives.** The goal of this paper is to investigate the stochastic behavior of the reliability model developed by Shu and Flowers (1998). This paper studies the steady-state behavior of the model, and derives expressions for reliability indices at steady state. An example is presented to illustrate application of the model to a relevant industry. Nonlinear regression analysis to estimate time to steady state for a range of parameters for the Weibull distribution is performed by Jiang (1999).

Factors that determine the reliability of repairable systems include failure characteristics of the part or component, system configuration, and repair strategy. Herein, the repair strategy is assumed to be a scheduled replacement, where the time interval between scheduled replacements is fixed. Under this scheduled replacement strategy, two repair policies are considered:

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- Perfectly maintained policy, where the scheduled replacement interarrival time $\Delta t = 0$. In this case, the population is under continuous supervision, such that when a part fails, the failure is immediately detected and the part replaced; and,
- Discretely maintained policy, where the scheduled replacement interarrival time $\Delta t > 0$. Here, maintenance for the population takes place every $\Delta t$ time units, such that when a part fails, it is not replaced until the next maintenance time.

For systems under these two repair policies, two issues are of interest:

- Theoretical analysis of the stabilization of the model, and
- Steady-state values of the reliability indices.

## 2 Reliability Model

This section first presents the terminology and notation used in the reliability model for repairable systems subject to system modifications. Next described are the assumptions of the model and the probability distribution used in simulations. Finally, some simulation results of the model are presented.

**2.1 Terminology and Notation.**

- *Part.* An item which is not subject to disassembly and is discarded or refurbished upon failure.
- *Socket.* In a system, a circuit or equipment position which holds a part of a given type.
- *System population.* A collection of systems with the same combination of parts.
- *Part population.* A collection of parts that hold the same socket position in the system.

Herein, the definition of a part is different from the one proposed by Ascher and Feingold (1984), where the *part* is discarded the first time that it fails, not allowing for the possibility of refurbishment.

1. Terms defined for parts
   - $\mathbf{x}$ age, measured in time, a real variable
   - $\mathbf{f(x)}$ failure density function, $\int_0^\infty f(x)dx = 1$
   - $\mathbf{i(x)}$ failure probability function, $i(x) = \int_0^x f(x)dx$
   - $\mathbf{r(x)}$ survival probability function, $r(x) = 1 - i(x) = \int_x^\infty f(x)dx$
   - $\mu$ mean life, $\mu = \int_0^\infty xf(x)dx = \int_0^\infty r(x)dx$
2. Terms defined for populations of parts
   - $\mathbf{t}$ time elapsed from when the population was first put into service
   - $\Delta\mathbf{t}$ interarrival time between scheduled replacements, where the time required for the actual maintenance activity is not included.
   - $\boldsymbol{\nu}(\mathbf{t})$ at time $\mathbf{t}$, replacement rate of the population, i.e., the fraction of the population being replaced per unit time
   - $\mathbf{t_n}$ time measured in units of $\Delta\mathbf{t}$, where $\mathbf{n}$ is a non-negative integer defined as $\mathbf{n} = \mathbf{t_n}/\Delta\mathbf{t}$
   - $\mathbf{q_i(t_n)}$ age distributions, i.e., at time $\mathbf{t_n}$, the fraction of parts with age $\mathbf{i}\Delta\mathbf{t}$, $\mathbf{i} = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}$
   - $\bar{a}(t)$ at time $\mathbf{t}$, average age of the population
   - $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$ replacement process of a constant-size population, where $\mathbf{f_1(x)}$ is the failure density function of original parts, and $\mathbf{f_2(x)}$ is the failure density function of replacement parts
3. Terms defined for systems
   - $\mathbf{m}$ number of sockets in a series system
   - $\mathbf{M}$ number of systems under observation

**2.2 Assumptions.** The reliability model developed by Shu and Flowers (1998) describes a population of $\mathbf{m}$-socket series systems. The population of $\mathbf{M}$ systems is represented as a collection of $\mathbf{m}$ populations of constituent parts, and parts are treated as members of their respective populations. Assumptions of the model are:

1. The size of the system population is constant.
2. The $\mathbf{m}$ parts have independent and possibly different failure density functions.
3. Failed parts can be replaced by components of the same or a different type. The remaining system either remains unchanged or is reconfigured to accommodate the replacement part.

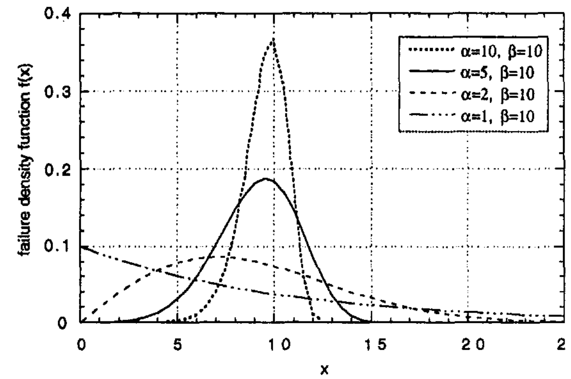

**Fig. 1 Weibull probability density functions**

The relationships of these assumptions to actual populations are as follows. Actual population sizes may not be constant. For example, some systems may not be replaced after failure, while other systems may be added after the initial population was put into service. The current model assumes that each failed system is replaced and that no additional systems are added. An extension of this model to accommodate non-constant-size populations is presented in Jiang et al. (1999). The second assumption allows each of the systems' constituent parts to have different failure density functions, thus making the model applicable to more systems than the many models that assume that all constituent parts have the same failure density function. However, the failure characteristics of the constituent parts are assumed to be independent, i.e., failure of one part does not affect failure of other parts in the system. Finally, most models assume that replacement parts have the same failure density function as the original parts. A feature distinguishing this model from others is that replacement parts can have failure density functions different from the original parts, which more accurately represents remanufacture and some maintenance activities.

**2.3 Failure Density Function.** Here, the two-parameter Weibull distribution is chosen to represent the various failure density functions of parts. The extension to use other distributions is fairly straightforward. The form of the Weibull probability density function is:

$$f(x) = \begin{cases} \dfrac{\alpha x^{\alpha-1}}{\beta^\alpha} \exp\left[-\left(\dfrac{x}{\beta}\right)^\alpha\right], & 0 \le x \le \infty \\ 0, & \text{elsewhere,} \end{cases} \tag{1}$$

where $\alpha > 0$, and $\beta > 0$. Figure 1 shows Weibull density functions corresponding to $\beta = 10$ paired with $\alpha = 1, 2, 5$, and $10$.

**2.4 Simulation.** At time $\mathbf{0}$, the failure density function of parts in the population is denoted $\mathbf{f_1(x)}$. The failure density function of parts that are used as replacements is denoted $\mathbf{f_2(x)}$. Thus, the replacement process of a constant-size population of parts is denoted $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$. If $\mathbf{f_2(x)}$ is different from $\mathbf{f_1(x)}$, system modification is performed in the replacement process.

At time $\mathbf{t_n} = \mathbf{n}\Delta\mathbf{t}$, the fractions of parts in the population whose ages equal $\mathbf{i}\Delta\mathbf{t}$, $\mathbf{i} = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}$, are:

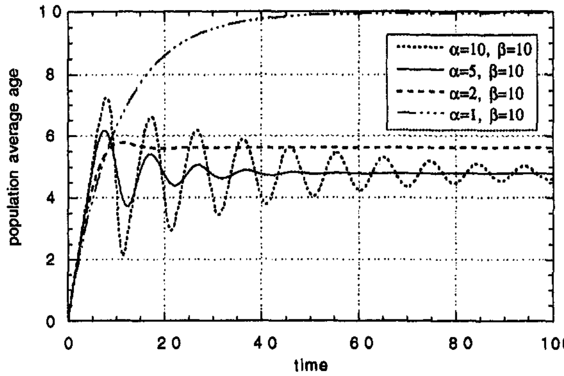


Fig. 2 Population average age (Δt = 0.1)

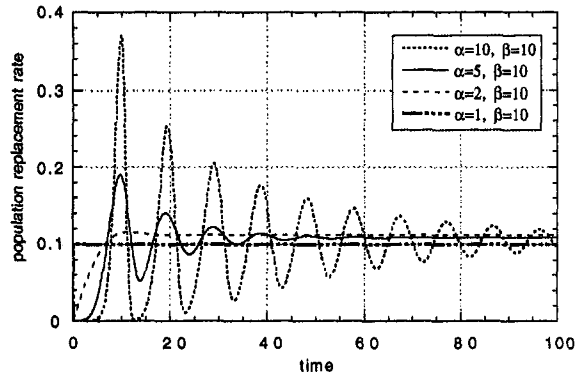


Fig. 3 Population replacement rate (Δt = 0.1)

$$q_n(t_n) = q_0(t_0)\left[1 - \int_0^{n\Delta t} f_1(x)dx\right]$$

$$q_{n-1}(t_n) = q_0(t_1)\left[1 - \int_0^{(n-1)\Delta t} f_2(x)dx\right]$$

$$\vdots$$

$$q_2(t_n) = q_0(t_{n-2})\left[1 - \int_0^{2\Delta t} f_2(x)dx\right]$$

$$q_1(t_n) = q_0(t_{n-1})\left[1 - \int_0^{\Delta t} f_2(x)dx\right]$$

$$q_0(t_n) = q_0(t_0)\left[\int_{(n-1)\Delta t}^{n\Delta t} f_1(x)dx\right] + q_0(t_1)\left[\int_{(n-2)\Delta t}^{(n-1)\Delta t} f_2(x)dx\right] + \cdots + q_0(t_{n-2})\left[\int_{\Delta t}^{2\Delta t} f_2(x)dx\right] + q_0(t_{n-1})\left[\int_0^{\Delta t} f_2(x)dx\right]. \quad (2)$$

The average age of the population is calculated as:

$$\bar{a}(t_n) = \sum_{i=0}^{n} q_i(t_n) i\Delta t. \quad (3)$$

The last term of Eq. (2), $\mathbf{q_0(t_n)}$, represents the fraction of the population at time $\mathbf{t_n}$ with age **0**. These 0-age parts are replacements for failed parts. Replacement rate is defined as the quantity of 0-age parts per unit time.

**2.5 Results of Simulation without System Modification.** This section concludes by discussing the simulation results of the process that does not involve system modifications, $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_1(x)}\}$. In this process, constant-size populations of identical parts are replaced by parts of the same type upon failure.

Figures 2 and 3 show population average age and replacement rate as a function of time. In both figures, each curve represents a population of parts with a particular Weibull-distributed failure density function. These plots correspond to a constant value of $\beta = 10$ paired with $\alpha = 1, 2, 5$ and 10. In Fig. 2, both the horizontal and vertical axes have the same units of time as the scheduled replacement interarrival time $\mathbf{\Delta t}$, e.g., minutes, hours, or years. The relationship between the oscillatory behavior and Weibull parameters is explained by Shu and Flowers (1998).

The result of interest to this paper is that the population average age and replacement rate reach steady state values. This result is consistent with Drenick's Theorem (Drenick, 1960) for a perfectly maintained population of parts (one where each failed component is immediately detected and replaced with a new part of the same type). Drenick's Theorem states that if a population of parts is put into service at time $\mathbf{t} = \mathbf{0}$ and the population is perfectly maintained, as time goes on, at each socket, there develops an unending sequence of failures which constitutes a random process called a renewal process. A renewal process is a counting process for which the interarrival times are independent and identically distributed with an arbitrary distribution (Ross, 1970). In a renewal process, the components or systems are renewed in a sequence, and it is assumed in the process that each renewal restarts the counting process as new (Ansell and Phillips, 1994). It is stated that such a population will, after an initial transient period, behave in the limit ($\mathbf{t} \to \infty$) like an ordinary Poisson process with exponential interarrival times, such as the case with components with a constant hazard rate (Grosh, 1975). The average number of renewals per unit time, i.e., the replacement rate, is known to converge to $1/\mu_1$, where $\mu_1$ is the mean life corresponding to $\mathbf{f_1(x)}$ (Blumenthal et al., 1971).

For the process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, where failed parts are replaced by parts with a different failure density function, of interest is how system modification affects the behavior of the model. Henceforth, the population is always assumed to be under system modification. In the next two sections, the replacement rate and average age are theoretically proven to reach steady state.

## 3 Replacement Rate

This section first presents the theoretic proof that the replacement rate for $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$ under a perfectly maintained repair policy reaches steady state. The result is then extended to the process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$ under a discretely maintained policy. The replacement rate function is expressed in the form of a convolution integral, and expressions of replacement rate at steady state are obtained.

**3.1 Perfectly Maintained Population.** A population under the perfectly maintained policy, where the scheduled replacement interarrival time $\mathbf{\Delta t} = \mathbf{0}$, is called a perfectly maintained population. The stabilization of such a population is studied below. In general, a given type of parts would have a typical maximum life where seldomly can parts survive past this maximum life. Then, the failure density function $\mathbf{f(x)}$ has the following property:

**Lemma 1.** Given a failure density function $\mathbf{f(x)}$, for any given small positive number $\delta$, there exists a time value $\mathbf{x^f}$ that yields $\int_{x^f}^{\infty} f(x)dx \leq \delta$.

Based on Lemma 1, if $\delta$ is of negligible value, for $\mathbf{x} \geq \mathbf{x^f}$, the survival probability $r(x) = \int_x^{\infty} f(x)dx \cong 0$. Since the value of $\mathbf{f(x)}$ is always positive, for $\mathbf{x} \geq \mathbf{x^f}$, the value of failure density function

$\mathbf{f(x)} \cong \mathbf{0}$. Time $\mathbf{x}^f$ is therefore the practical maximum life span for parts with failure density function $\mathbf{f(x)}$.

**Lemma 2.** Given the failure density functions $\mathbf{f_1(x)}$ and $\mathbf{f_2(x)}$, for a perfectly maintained process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, the replacement rate function is

$$\nu(t) = \int_0^t \nu(t-x) f_2(x)dx + f_1(t) - f_2(t) + u_1(t), \quad (4)$$

where $\mathbf{t} \geq \mathbf{0}$ and $\mathbf{u_1(t)}$ is a unit impulse at $\mathbf{t} = \mathbf{0}$.

*Proof:* For a discretely maintained process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, at time $\mathbf{t_0} = \mathbf{0}$, since all the parts in the population have just been put into service, the fraction of parts with age $\mathbf{0}$, $\mathbf{q_0(t_0)} = \mathbf{1}$.

At time $\mathbf{t_n} = \mathbf{n\Delta t}$, $\mathbf{n} > \mathbf{0}$, according to Eq. (2), the fraction of parts with age $\mathbf{0}$, i.e., parts being replaced at time $\mathbf{t_n}$, is

$$q_0(t_n) = q_0(t_0)\left[\int_{(n-1)\Delta t}^{n\Delta t} f_1(x)dx\right] + q_0(t_1)\left[\int_{(n-2)\Delta t}^{(n-1)\Delta t} f_2(x)dx\right]$$

$$+ \cdots + q_0(t_{n-2})\left[\int_{\Delta t}^{2\Delta t} f_2(x)dx\right] + q_0(t_{n-1})\left[\int_0^{\Delta t} f_2(x)dx\right]$$

$$= \sum_{j=1}^{n-1} q_0(t_{n-j}) \int_{(j-1)\Delta t}^{j\Delta t} f_2(x)dx + q_0(t_0)\int_{(n-1)\Delta t}^{n\Delta t} f_1(x)dx$$

$$= \sum_{j=1}^{n} q_0(t_{n-j}) \int_{(j-1)\Delta t}^{j\Delta t} f_2(x)dx + q_0(t_0)\int_{(n-1)\Delta t}^{n\Delta t} (f_1(x) - f_2(x))dx. \quad (5)$$

By definition, the population replacement rate is the fraction of parts being replaced per unit time, i.e., $\nu((n-j)\Delta t) = q_0(t_{n-j})/\Delta t$. At time $\mathbf{t_n} = \mathbf{n\Delta t}$, $\mathbf{n} > \mathbf{0}$, the replacement rate $\boldsymbol{\nu}(\mathbf{n\Delta t})$ is

$$\nu(n\Delta t) = \frac{q_0(t_n)}{\Delta t} = \sum_{j=1}^{n} \nu((n-j)\Delta t) \int_{(j-1)\Delta t}^{j\Delta t} f_2(x)dx + \frac{q_0(t_0)}{\Delta t}\left(\int_{(n-1)\Delta t}^{n\Delta t} (f_1(x) - f_2(x))dx\right). \quad (6)$$

For a perfectly maintained process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, at the limit $\mathbf{\Delta t} \to \mathbf{0}$, $\mathbf{\Delta t} = \mathbf{dx}$, then

$$\int_{(j-1)\Delta t}^{j\Delta t} f_2(x)dx = f_2(j\Delta t)\Delta t \quad (7)$$

and

$$\int_{(n-1)\Delta t}^{n\Delta t} (f_1(x) - f_2(x))dx = (f_1(n\Delta t) - f_2(n\Delta t))\Delta t. \quad (8)$$

Substitution of Eqs. (7) and (8) into Eq. (6) gives

$$\nu(n\Delta t) = \sum_{j=1}^{n} \nu((n-j)\Delta t) f_2(j\Delta t)\Delta t + q_0(t_0)(f_1(n\Delta t) - f_2(n\Delta t)). \quad (9)$$

Note that $\mathbf{q_0(t_0)} = \mathbf{1}$. Let $\mathbf{x} = \mathbf{j\Delta t}$ and $\mathbf{t} = \mathbf{t_n} = \mathbf{n\Delta t}$, $\mathbf{n} > \mathbf{0}$, then, $\mathbf{dx} = \mathbf{\Delta t}$. At the limit $\mathbf{\Delta t} \to \mathbf{0}$, the preceding summation becomes the integral

$$\nu(t) = \int_0^t \nu(t-x) f_2(x)dx + f_1(t) - f_2(t). \quad (10)$$

The above equation does not include the point $\boldsymbol{\nu}(\mathbf{0})$. At time $\mathbf{0}$, the entire population of parts has just been put into service. It is assumed that the time needed to complete this activity is $\mathbf{0}$. Hence, the replacement rate is $\boldsymbol{\nu}(\mathbf{0}) = \mathbf{q_0(t_0)/0} = \mathbf{1/0}$, which can be represented as a unit impulse $\mathbf{u_1(t)}$. Then, the replacement rate function $\mathbf{v(t)}$ is consistent with Eq. (4).

**Property 1.** Given failure density functions $\mathbf{f_1(x)}$ and $\mathbf{f_2(x)}$, for a perfectly maintained process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, the replacement rate will reach a constant value $\mathbf{v_s}$ and

$$\nu_s = \frac{1}{\mu_2}, \quad (11)$$

where $\mu_2$ is the mean life of parts with failure density function $\mathbf{f_2(x)}$.

*Proof:* Based on Lemma 1, let time $\mathbf{t_f}$ be defined as

$$t_f = \max(x_1^f, x_2^f), \quad (12)$$

where $\mathbf{x}_1^f$ and $\mathbf{x}_2^f$ are the practical maximum life spans for $\mathbf{f_1(x)}$ and $\mathbf{f_2(x)}$ respectively. Then, for $\mathbf{t} > \mathbf{t_f}$, values of failure density functions $\mathbf{f_1(t)} \cong \mathbf{0}$ and $\mathbf{f_2(t)} \cong \mathbf{0}$. The replacement rate function, i.e., Eq. (4), can be rewritten as

$$\nu(t) = \int_0^t \nu(t-x) f_2(x)dx + u_1(t). \quad (13)$$

The above equation is in the form of a *convolution integral.* To obtain the value of the replacement rate function at time $\mathbf{t}$, $\nu(\mathbf{t})$, it is necessary to integrate over all combinations of $\mathbf{f_2(t)}$ and $\boldsymbol{\nu}(\mathbf{t})$ from $\mathbf{0}$ to $\mathbf{t}$. Denoting the Laplace transform operator as $l$, the Laplace transform of the replacement rate function is

$$V(s) = l(\nu(t)) = \int_0^\infty \exp(-st)\left(\int_0^t \nu(t-x) f_2(x)dx\right)dt + \int_0^\infty u_1(t)\exp(-st)dt. \quad (14)$$

The Laplace transform of the unit impulse equals $\mathbf{1}$. Since the Laplace transform of a convolution of two functions is the product of the two functions (Raven, 1966), Eq. (14) can be rewritten as

$$V(s) = V(s)F_2(s) + 1, \quad (15)$$

where $F_2(s) = \int_0^\infty f_2(t)\exp(-st)dt$. The above equation can be rewritten as

$$V(s) = \frac{1}{1 - F_2(s)}. \quad (16)$$

Based on a property of the Laplace transform, we have the following equation:

$$F_2(s) = l(f_2(t)) = l\left(\frac{d}{dt} i_2(t)\right) = sI_2(s) - i_2(0) = sI_2(s), \quad (17)$$

where $\mathbf{i_2(t)}$ is the failure probability function of $\mathbf{f_2(t)}$. $\mathbf{I_2(s)}$ is the Laplace transform of the failure probability function $\mathbf{i_2(t)}$, which is

$$I_2(s) = l(i_2(t)) = l(1 - r_2(t)) = \frac{1}{s} - l(r_2(t)), \tag{18}$$

where $\mathbf{r_2(t)}$ is the survival probability function of $\mathbf{f_2(t)}$.

Substitution of Equation (18) into Eq. (17) gives

$$F_2(s) = 1 - sl(r_2(t)). \tag{19}$$

Then, Eq. (16) can be rewritten as

$$V(s) = \frac{1}{sl(r_2(t))}. \tag{20}$$

Based on the final-value theorem of Laplace transform, we have

$$\begin{aligned} \nu_s &= \nu(\infty) \\ &= \lim_{s\to 0} sV(s) \\ &= \lim_{s\to 0} \frac{1}{\int_0^\infty r_2(t)\exp(-st)dt} \\ &= \frac{1}{\int_0^\infty r_2(t)dt} = \frac{1}{\mu_2}. \end{aligned} \tag{21}$$

Hence, the replacement rate $\boldsymbol{\nu(t)}$ will eventually reach a constant value $\boldsymbol{\nu_s}$.

Property 1 indicates that in a perfectly maintained process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, the steady-state value for a replacement rate depends only on the failure density function $\mathbf{f_2(x)}$. This result is reasonable since as time increases to $\mathbf{x_1^f}$, the practical maximum life span of $\mathbf{f_1(x)}$, almost all the original parts in the population with a failure density function $\mathbf{f_1(x)}$ will have been replaced with parts with a failure density function $\mathbf{f_2(x)}$. After time $\mathbf{x_1^f}$, the perfectly maintained process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$ behaves in a similar way as the process $\{\mathbf{f_2(x)} \leftarrow \mathbf{f_2(x)}\}$, which is a Superimposed Renewal Process.

At the limit as $\boldsymbol{\Delta t} \to \infty$, a Superimposed Renewal Process behaves like a Homogeneous Poisson Process with a constant hazard rate, which can be represented by an exponential distribution. The following is the derivation for the exponential distribution used to represent the steady state of the Superimposed Renewal Process.

For a perfectly maintained process $\{\mathbf{f_2(x)} \leftarrow \mathbf{f_2(x)}\}$, suppose that the replaced parts have an exponential distribution with a failure density function given by

$$f_2(x) = \lambda \exp(-\lambda x), \quad x \geq 0, \tag{22}$$

where $\lambda$ is a positive constant. The Laplace transform of $\mathbf{f_2(x)}$ is

$$F_2(s) = \int_0^\infty \exp(-st)\lambda \exp(-\lambda t)dt = \frac{\lambda}{(\lambda + s)}. \tag{23}$$

Substitution of the above equation into Eq. (16) yields

$$V(s) = \frac{\lambda}{s} + 1. \tag{24}$$

The replacement rate function $\boldsymbol{\nu(t)}$, which is the inverse Laplace transform of $\mathbf{V(s)}$, can be obtained as

$$\nu(t) = \lambda + u_1(t), \quad t \geq 0. \tag{25}$$

Here, we find that when time $\mathbf{t} > \mathbf{0}$, the replacement rate of a process $\{\mathbf{f_2(x)} \leftarrow \mathbf{f_2(x)}\}$, where $\mathbf{f_2(x)}$ is an exponential distribution, is constant with value $\lambda$, which equals the reciprocal of the mean life of $\mathbf{f_2(x)}$. Since at the steady state of the process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, based on Property 1, the replacement rate is also constant with value $\mathbf{1}/\mu_2$, we can denote the failure density function for the population at steady state as $\mathbf{f_s(x)}$, where

$$f_s(x) = \frac{1}{\mu_2}\exp\left(-\frac{x}{\mu_2}\right), \quad x > 0, \tag{26}$$

which is in the form of an exponential distribution. Furthermore, the replacement rate of process $\{\mathbf{f_2(x)} \leftarrow \mathbf{f_2(x)}\}$ converges to $\mathbf{1}/\mu_2$, the reciprocal of the mean life of $\mathbf{f_s(x)}$. This result is consistent with Property 1.

**3.2 Discretely Maintained Population.** The perfectly maintained process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, is the result of a discretely maintained process in the limit $\boldsymbol{\Delta t} \to \mathbf{0}$. The following property for a discretely maintained process can be proven in a similar way as for a perfectly maintained process.

**Property 2.** Given the failure density functions $\mathbf{f_1(x)}$ and $\mathbf{f_2(x)}$ and scheduled replacement interarrival time $\boldsymbol{\Delta t}$, for a discretely maintained process $\{\mathbf{f_1(x)} \leftarrow \mathbf{f_2(x)}\}$, the replacement rate $\boldsymbol{\nu(n\Delta t)}$ will reach a constant value $\boldsymbol{\nu_s}$,

$$\nu_s = \frac{1}{\sum_{i=0}^{k_2^f - 1} r_2(i\Delta t)\Delta t}, \tag{27}$$

where $\mathbf{k_2^f} = \mathbf{x_2^f}/\boldsymbol{\Delta t}$ and $\mathbf{x_2^f}$ is the practical maximum life span for $\mathbf{f_2(x)}$.

At the limit, $\boldsymbol{\Delta t} \to \mathbf{0}$, the steady-state value for the replacement rate becomes

$$\lim_{\Delta t\to 0} \nu_s = \lim_{\Delta t\to 0} \frac{1}{\sum_{i=0}^{k_2^f - 1} r_2(i\Delta t)\Delta t} = \frac{1}{\mu_2}. \tag{28}$$

The effect of scheduled replacement interarrival time $\boldsymbol{\Delta t}$ on the steady-state replacement rate can be derived from Property 2. Let $\boldsymbol{\Delta t'} = \mathbf{y}\boldsymbol{\Delta t}$, $\mathbf{k'} = x_2^f/\boldsymbol{\Delta t'} = k_2^f/\mathbf{y}$, where $\mathbf{y}$ is a positive integer. Since the survival probability $\mathbf{r_2(t)}$ monotonically decreases, the following relationship exists:

$$\sum_{i=0}^{k'-1} r_2(i\Delta t') > \frac{1}{y}\sum_{i=0}^{k_2^f - 1} r_2(i\Delta t). \tag{29}$$

Then, the replacement rate is

$$\nu_s' = \frac{1}{\Delta t' \sum_{i=0}^{k'-1} r_2(i\Delta t')} < \frac{1}{\Delta t \sum_{i=0}^{k_2^f - 1} r_2(i\Delta t)} = \nu_s. \tag{30}$$

Hence, as interarrival time $\boldsymbol{\Delta t}$ increases, steady-state replacement rate decreases. Although the steady-state replacement rate decreases, the number of failed systems that are not in operation at any given time may be higher, because the time between repair activities is longer.

## 4 Average Age and Age Distributions

This section presents the derivation of steady-state expressions for population average age and age distributions. Since the perfectly maintained process $\{f_1(x) \leftarrow f_2(x)\}$ is the result of a discretely maintained process at the limit, $\Delta t \rightarrow 0$, we begin by investigating the discretely maintained process.

### 4.1 Discretely Maintained Population.

**Property 3.** Given the failure density functions $f_1(x)$ and $f_2(x)$ and scheduled replacement interarrival time $\Delta t$, for a discretely maintained process $\{f_1(x) \leftarrow f_2(x)\}$, at steady state, where time is equal to $n\Delta t$, the fractions of age distributions are

$$q_i(t_n) = \begin{cases} \dfrac{1}{\sum_{j=0}^{k_2^f-1} r_2(j\Delta t)} r_2(i\Delta t) & 0 \le i < k_2^f \\ 0 & k_2^f \le i \le n. \end{cases} \quad (31)$$

where $k_2^f = x_2^f/\Delta t$ and $x_2^f$ is the practical maximum life span for $f_2(x)$.

*Proof:* First, let $t_s = k_s\Delta t$, where $k_s$ is a positive integer, to be a time at which the replacement rate, $\nu(t)$, has reached steady state. A necessary condition on $t_n = n\Delta t$, the time for which we are calculating age distribution fractions, is that $t_n$ is larger than $t_s$ by at least the interval $x_2^f$, the maximum practical life span corresponding to $f_2(x)$, i.e., $n\Delta t \ge (k_s + k_2^f)\Delta t$. Also, $t_s$ is always greater than the maximum of the practical maximum life spans corresponding to $f_1(x)$ and $f_2(x)$, i.e., $k_s > \max(x_1^f/\Delta t, x_2^f/\Delta t)$.

At time $t_n = n\Delta t$, according to Eq. (2), the age distributions are

$$q_i(t_n) = q_0(t_{n-i})\left[1 - \int_0^{i\Delta t} f_2(x)dx\right], \quad i = 0, \ldots, n-1 \quad (32)$$

and

$$q_n(t_n) = q_0(t_0)\left[1 - \int_0^{n\Delta t} f_1(x)dx\right], \quad (33)$$

where $q_i(t_n)$ is the fraction of parts with age $i\Delta t$.

Based on Lemma 1, the survival probability

$$1 - \int_0^{i\Delta t} f_2(x)dx = 0, \quad i = k_2^f, k_2^f + 1, \ldots, n-1 \quad (34)$$

and

$$1 - \int_0^{n\Delta t} f_1(x)dx = 0, \quad (35)$$

which implies that the fraction of parts

$$q_i(t_n) = 0, \quad i = k_2^f, k_2^f + 1, \ldots, n. \quad (36)$$

To calculate $q_0(t_{n-i})$, needed to find $q_i(t_n)$, $0 \le i < k_2^f$, the value of the replacement rate from time $(n - k_2^f)\Delta t$ to time $n\Delta t$ is needed. Note that the replacement rate reaches steady state after time $t_s = k_s\Delta t$, as per the definition for $t_s$. From the condition on $t_n$, $(n - k_2^f)\Delta t \ge k_s\Delta t$. Therefore, the replacement rate has a constant value from time $(n - k_2^f)\Delta t$ to time $n\Delta t$. Based on Property 2, the steady-state replacement rate is given by $\nu_s = 1/[\sum_{j=0}^{k_2^f-1} r_2(j\Delta t)\Delta t]$. Thus, the fraction of parts that fail during interarrival time $\Delta t$ is also constant from time $(n - k_2^f)\Delta t$ to time $n\Delta t$ and can be expressed as

$$q_0(t_{n-i}) = \nu_s\Delta t = \frac{1}{\sum_{j=0}^{k_2^f-1} r_2(j\Delta t)}, \quad 0 \le i < k_2^f. \quad (37)$$

Substitution of Eq. (37) into Eq. (32) yields:

$$q_i(t_n) = \frac{1}{\sum_{j=0}^{k_2^f-1} r_2(j\Delta t)} r_2(i\Delta t), \quad 0 \le i < k_2^f. \quad (38)$$

Combination of Eqs. (36) and (38) yields Eq. (31).

At steady state, the average age is

$$\bar{a}_s = \frac{1}{\sum_{j=0}^{k_2^f-1} r_2(j\Delta t)} \sum_{i=0}^{k_2^f-1} r_2(i\Delta t)i\Delta t. \quad (39)$$

Property 3 enables direct calculation of the age distribution at steady state without performing a simulation.

**4.2 Perfectly Maintained Population.** The replacement rate at steady state has been shown to be $\nu_s = 1/\mu_2$, where $\mu_2$ is the mean of $f_2(x)$.

All the expressions of reliability indices for a perfectly maintained process can be derived in a similar way as for a discretely maintained process. The age distribution function $h(x)$ is defined as

$$h(x) = \frac{r_2(x)}{\mu_2}, \quad (40)$$

then, the fraction of parts aged from $x_1$ to $x_2$, denoted as $q(x)$ is

$$q(x) = \int_{x_1}^{x_2} h(x)dx. \quad (41)$$

The population average age at steady state is

$$\bar{a}_s = \frac{\int_0^{\infty} r_2(x)xdx}{\mu_2}. \quad (42)$$

## 5 Example Simulation and Discussion

Preliminary experimental validation of this model was performed by Shu and Flowers (1998). Also, hypothetical populations of mechanical systems were studied using the model to examine tradeoffs between component reliability and cost over the life cycle of the populations. Jiang et al. (1999) further developed and applied the model to failure data of actual systems.

This section presents an example to illustrate application of this model to one of the many relevant remanufacturing industries. A compilation by Lund (1996) of 9,903 remanufacturers identified the most dominant product sectors as automotive, electrical apparatus, tires, and toner cartridges. An example in the tire retreading industry is chosen because the parts and processes involved are fairly simple and can be briefly explained to a sufficient level.

First, statistics on the tire retread industry and a description of the corresponding remanufacture processes are provided. Next provided is background for reasonable estimation of the Weibull functions that could describe a population of truck-tire treads. Simulations of average age, replacement rate, and steady-state age distributions are then performed using these hypothetical Weibull

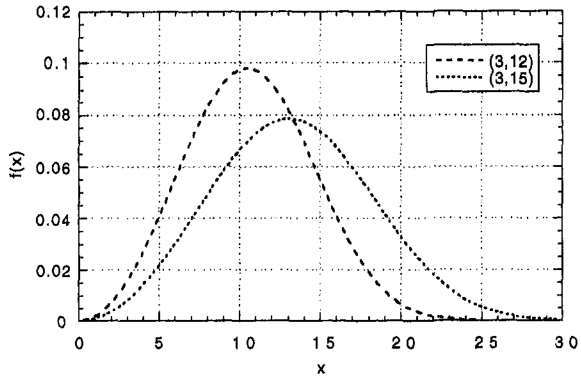


**Fig. 4 Weibull failure density functions**

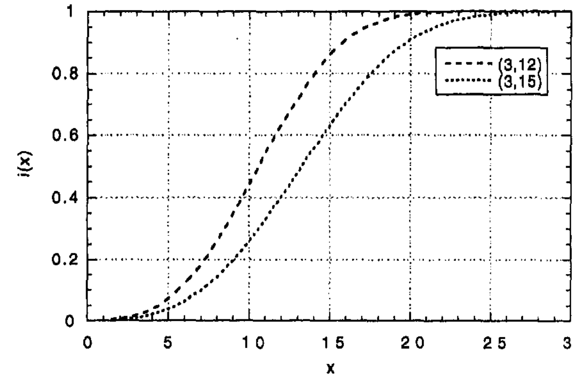


**Fig. 5 Weibull probabilities of failure**

functions. Also discussed is the physical significance of the simulations.

**5.1 Tire Retreading Background.** According to the Tire Retread Information Bureau, approximately 30.9 million retreaded tires were sold in the U.S. and Canada in 1997, corresponding to sales totaling more than 2 billion dollars. Of the 30.9 million, approximately 12% were passenger car tires, 25% light truck tires, 60% medium and heavy truck tires, and 3% others, including tires for aircraft, off-the-road vehicles, motorcycles, farm equipment, etc. To manufacture a new truck tire requires 22 gallons of oil, compared to the 7 gallons required for retread. The cost of a retreaded tire is 30% to 50% less than the cost of a new tire, yet the reliability of a properly retreaded tire may be comparable to that of a new tire.

A typical retread process begins with a visual inspection to determine which tire casings can be retreaded. Next, buffing is performed, usually on a lathe-type machine, to remove previous tread material and to shape, size and texture the casing surface to receive a new tread. Repairs for damages such as nail holes are made at this time. Further inspection is then performed to detect damage such as pinhole leaks and separations. Once repairs on salvageable tires are made, the casing is sprayed with adhesive cement. In a pre-cure system, a layer of cushion gum is added to the back of the tread, which is then fitted and centered on the casing. The casing and tread assembly is then heated inside a curing chamber, bonding the tread to the casing. The finished retread is inspected for flaws and proper bonding, and then trimmed of overflow and painted.

**5.2 Estimation of Weibull Parameters.** Truck tires, the majority of retreads, can be retreaded up to three or four times. Retreading is recommended after approximately 200,000 miles, or when the tread wears to 6/32″. Under certain operating conditions, a conservative estimate for the time between tire retreads is 12 months. However, the wear rate and retreadability is dependant on a large number of factors, including vehicle and tire maintenance, loading, application, and environmental exposure. Therefore, the precision of estimated time between removals is not high.

Suppose a population of tires whose tread life is represented by a Weibull distribution, $\boldsymbol{\alpha} = \mathbf{3}$, $\boldsymbol{\beta} = \mathbf{12}$, is installed. Recall that $\beta$ approximates the expected time of 12 months between removal for retreading, and $\alpha$ is proportional to the precision of the expected removal time. Suppose further that a retreading technology is then developed that can lengthen the expected time between removals to 15 months.

Below, $(\alpha, \beta)$ is used to denote a Weibull failure density function. For example, **(3, 12)** represents a Weibull distribution with $\boldsymbol{\alpha} = \mathbf{3}$, $\boldsymbol{\beta} = \mathbf{12}$. Figure 4 shows the hypothetical Weibull distributions corresponding to the treads on the originally installed tires and the improved retreads. Figure 5 shows the probabilities of failure corresponding to the Weibull distributions of Fig. 4.

**5.3 Simulation Results and Discussion.** Using the model developed in this paper, simulations for the processes $\{\mathbf{(3, 12)} \leftarrow \mathbf{(3, 12)}\}$, $\{\mathbf{(3, 12)} \leftarrow \mathbf{(3, 15)}\}$, and $\{\mathbf{(3, 15)} \leftarrow \mathbf{(3, 15)}\}$ were performed. The process $\{\mathbf{(3, 12)} \leftarrow \mathbf{(3, 12)}\}$ describes an initial population of tire treads that become worn after approximately 12 months of use, and are replaced as required by treads with the same life expectancy. The process $\{\mathbf{(3, 12)} \leftarrow \mathbf{(3, 15)}\}$ represents the case where an initial population of treads with a 12-month life expectancy are replaced when worn by treads with a 15-month life expectancy. The process $\{\mathbf{(3, 15)} \leftarrow \mathbf{(3, 15)}\}$ is shown for comparison purposes.

Figure 6 shows the average ages of the population of treads. Figure 7 shows the replacement rates for the population. Physically, the replacement rate represents the fraction of the tread population that is removed for replacement as a function of time. Figures 6 and 7 illustrate that as the average age of the population increases, an increasing portion of the population is removed and replaced with 0-age treads. The installation of the 0-age treads then

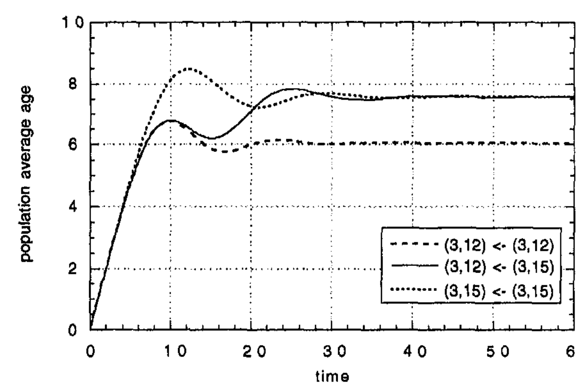


**Fig. 6 Population average ages**

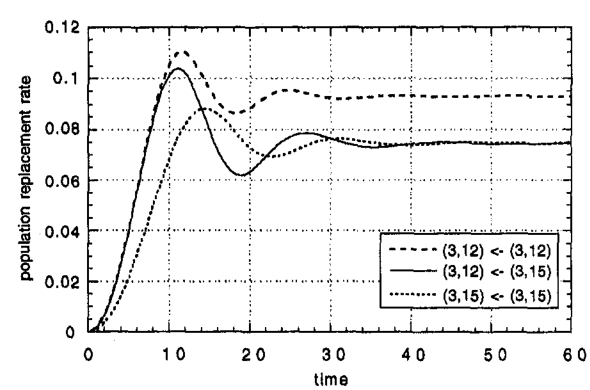


**Fig. 7 Population replacement rates**

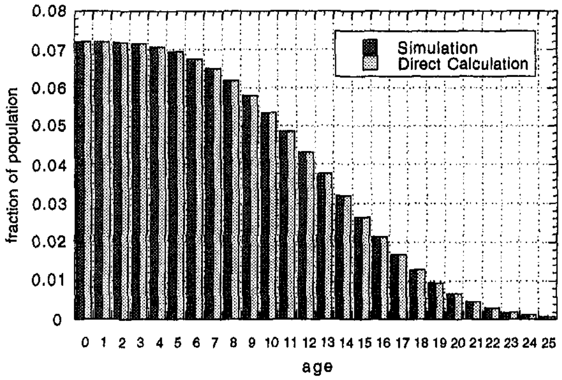


**Fig. 8 Steady state age distributions {(3, 12) ← (3, 15)}**

reduces the average age and consequently, the replacement rate. Figures 6 and 7 show that both average age and replacement rate for all processes reach steady state.

Finally, the age distributions at steady state are calculated for the process **{(3, 12) ← (3, 15)}** using both direct calculation based on Property 3 and simulation. Figure 8 shows that $\mathbf{q}_i(\mathbf{t}_s)$, the steady-state age distributions calculated directly agree with those obtained using simulation.

For a relatively low value of $\alpha$, as is the case in this example, the certainty that parts require replacement exactly at age $\beta$ is correspondingly low, as shown in Figs. 1 and 4. Therefore, there is spread in the time at which members of the original population are replaced. This dispersed replacement process leads to a population with a variety of part ages, which then further disperses the time to failure of individual members of the population. This repeated diffusion in both average age and replacement rate leads these processes to steady state. Furthermore, the steady-state values of the processes **{(3, 12) ← (3, 15)}** and **{(3, 15) ← (3, 15)}** are identical, supporting the theoretic analysis performed earlier.

Figure 6 shows that the steady-state average age of each process is approximately one half of the $\beta$ of the corresponding replacement part. For a population of parts with an expected life of approximately $\boldsymbol{\beta} = \mathbf{15}$, Fig. 8 shows that at steady state, the number of parts with age less than $\beta$ is greater than the number of parts with age greater than $\beta$. This is increasingly true as $\alpha$ increases, where the steady-state average age approaches $\boldsymbol{\beta}/\mathbf{2}$. In the limit $\alpha \to \infty$, all members of a population, will upon installation, have an identical, increasing age until precisely time and age $\beta$, when all parts will fail. The entire population is then replaced with 0-age parts that repeat the cycle. The average age of such a population never reaches steady state but is bound between **0** and $\beta$.

Consistent with intuition is that the steady-state replacement rate of a population with a lower expected life will be higher than a population with a higher expected life. Figure 7 shows that the steady-state replacement rate of a population that continued with the **(3, 12)** treads is approximately 9.5%, while that of a population that switched over to the **(3, 15)** treads is approximately 7.5%.

Figures 6 through 8 can be used to estimate replacement requirements as a function of time for a population of parts. Both suppliers and buyers could use this estimation to plan production or purchasing requirements to support an existing population of parts. Furthermore, this information is useful to system designers who may perform tradeoff analyses between life-cycle reliability, cost, and replacement requirements.

## 6 Summary

In remanufacture, modifications may be made to systems since replacement parts can be of the same type as, or of a different type from, the original parts. It follows that the failure density functions of the replacement parts may be different from those of the original parts. Most existing reliability models of repairable systems do not allow for the possibility of system modifications.

This paper investigates the stochastic behavior of a reliability model for repairable systems subject to system modifications. This model can be used for estimation of life-cycle replacement requirements for systems that are remanufactured, thereby facilitating decisions during system design and use.

Two different repair policies were considered: the perfectly maintained, where failed parts are replaced as soon as the failure occurs, and the discretely maintained, where replacements are made at predetermined times. Under both policies, the theoretical analyses performed support simulation results showing that population average age and replacement rate reach steady state. During steady state, the model behaves like a Homogeneous Poisson Process with a constant replacement rate. Explicit expressions of reliability indices of the system at steady state are obtained. Finally, an example in a relevant industry illustrates application of the model studied in this paper.

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